

Interactions in QFT & Feynman Rules

Now that we've quantized the free scalar field, we'll consider quantum field theories with interactions.

Our goal will be to compute scattering amplitudes, in analogy with our study of ~~the~~ perturbation theory in quantum mechanics.

Relatively soon, we'll see that these amplitudes can be expressed via Feynman diagrams which will give us a fast algorithm to computing them.

Just as in time-dependent perturbation theory in QM where the perturbing Hamiltonian allowed transitions between eigenstates of the unperturbed Hamiltonian, the interactions in QFT will allow for transitions between multiparticle states of the free theory — decays & scattering.

We'll introduce machinery to compute these amplitudes & convert them into measurable quantities like cross sections & decay rates.

ϕ^4 theory

Building off our free particle example, consider the real scalar field theory before, with a new quartic term in its potential energy:

$$\propto \lambda \phi^4$$

The Lagrangian is:

$$\mathcal{L} = \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} m^2 \phi^2 - \frac{1}{4!} \lambda \phi^4$$

The factor $4!$ is introduced for later convenience. The prefactor λ is a dimensionless coupling constant.

The Hamiltonian is

$$H = \int d^3x \mathcal{H} = \int d^3x \left[\frac{1}{2} \dot{\phi}^2 + \frac{1}{2} (\vec{\nabla} \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} \phi^4 \right]$$

As in QM, our first step is to identify the unperturbed Hamiltonian, H_0 :

$$H_0 = \int d^3x \left[\frac{1}{2} \dot{\phi}^2 + \frac{1}{2} (\vec{\nabla} \phi)^2 + \frac{1}{2} m^2 \phi^2 \right]$$

and the interacting part:

$$H_{\text{int}} = \int d^3x \frac{\lambda}{4!} \phi^4$$

A First Look at Scattering

Just as in QM, we can work in the small λ limit, and look for transitions between the free particle states that are eigenstates of H_0 .

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To set up the idea a bit more, consider the scattering process:

$$\phi(p_1) + \phi(p_2) \rightarrow \phi(p_3) + \phi(p_4)$$

Just as in QM, we want to compute a transition amplitude:

$$\langle f | \left(-i \int dt H_{int} \right) | i \rangle$$

where, $|i\rangle$, $|f\rangle$ are the initial/final states. For our example process, they are of the form:

$$|i\rangle = |\vec{p}_1, \vec{p}_2\rangle = \sqrt{2E_{\vec{p}_1}} \sqrt{2E_{\vec{p}_2}} a_{\vec{p}_1}^+ a_{\vec{p}_2}^+ |0\rangle$$

$$\langle f| = \langle \vec{p}_3, \vec{p}_4| = \sqrt{2E_{\vec{p}_3}} \sqrt{2E_{\vec{p}_4}} \langle 0| a_{\vec{p}_3} a_{\vec{p}_4}$$

With H_{int} defined above, the amplitude of interest is thus:

$$\langle \vec{p}_3, \vec{p}_4 | \underbrace{\left\{ -i \int dt \int d^3x \frac{\lambda}{4!} \phi^4(t, x) \right\}}_{= \int d^4x} | \vec{p}_1, \vec{p}_2 \rangle$$

Since we're working in perturbation theory, we ~~do not~~ construct $\phi(x)$ explicitly in terms of the ladder operators.

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(a_{\vec{p}} e^{-ip \cdot x} + a_{\vec{p}}^\dagger e^{ip \cdot x} \right)$$

To start evaluating this, we consider the action of $\phi(x)$ on $|\vec{p}\rangle$:

$$\phi(x) |\vec{p}\rangle = \left(\int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{k}}}} a_{\vec{k}} e^{-ik \cdot x} + \dots \right) \left(\sqrt{2E_{\vec{p}}} a_{\vec{p}}^\dagger |0\rangle \right)$$

$$= \int \frac{d^3k}{(2\pi)^3} \sqrt{\frac{2E_{\vec{p}}}{2E_{\vec{k}}}} e^{-ik \cdot x} \left(\underbrace{[a_{\vec{k}}, a_{\vec{p}}^\dagger]}_{(2\pi)^3 \delta^3(\vec{k} - \vec{p})} - \cancel{a_{\vec{p}}^\dagger a_{\vec{k}}} \right) |0\rangle$$

$$= \int d^3k e^{-ik \cdot x} \delta^3(\vec{k} - \vec{p}) |0\rangle = e^{-ip \cdot x} |0\rangle$$

Similarly, the action of ϕ on $\langle \vec{p} |$ is:

$$\langle \vec{p} | \phi(x) = \langle 0 | e^{ip \cdot x}$$

These types of operations are called contractions of the field operator w/ the ~~field~~ external state.

We'll use the notation

$$\overline{\phi(x) |\vec{p}\rangle} \quad \text{and} \quad \overline{\langle \vec{p} | \phi(x)}$$

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This is useful because the external states are unordered, and we can see all the pairings quickly!

We can contract the four field operators w/ the four final states. For example:

$$\langle \vec{p}_3 \vec{p}_4 | \phi \phi \phi \phi | \vec{p}_1 \vec{p}_2 \rangle$$

or

$$\langle \vec{p}_3 \vec{p}_4 | \phi \phi \phi \phi | \vec{p}_1 \vec{p}_2 \rangle$$

Some quick counting reveals there are $4!$ possible contractions, all leading to the same result.

The transition amplitude is thus:

$$\langle \vec{p}_3 \vec{p}_4 | \left(-i \int d^4x \frac{\lambda}{4!} \phi^4(x) \right) | \vec{p}_1 \vec{p}_2 \rangle$$

$$= -i \frac{\lambda}{4!} \times 4! \times \int d^4x \langle \vec{p}_3 \vec{p}_4 | \phi \phi \phi \phi | \vec{p}_1 \vec{p}_2 \rangle$$

$$= -i \lambda \cdot \int d^4x e^{-i(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4) \cdot x}$$

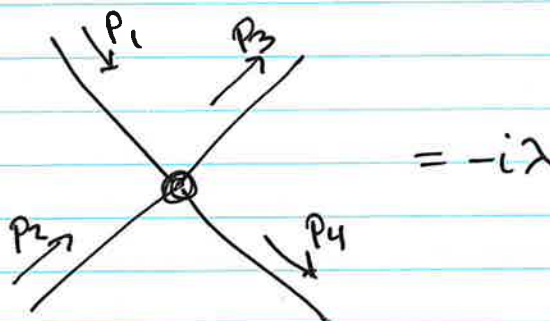
$$= -i \lambda (2\pi)^4 \delta^4(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4)$$

→ this is the transition amplitude!

Note that the transition amplitude includes a δ -fcn. that enforces energy & momentum conservation.

The amplitude can be represented graphically in terms of a

Feynman diagram:



the association w/ a factor $-i\lambda$ is the Feynman rule (in momentum space) we will learn to associate w/ this diagram shortly.

Later, we'll return to how this amplitude can be used to compute a cross-section for $\phi\phi \rightarrow \phi\phi$ scattering.

First, we'll address several issues we swept under the rug:

→ We used H_0 for the initial & final states, and the vacuum and the mode expansion of the free field in evaluating the matrix element.

→ How do we rigorously justify this in QFT?

Perturbative Calculation of Correlation Functions

Before computing matrix elements between multi-particle states, we'll consider simpler objects — correlation functions, or vacuum-to-vacuum transitions.

Consider the two-point correlation function:

$$\langle \Omega | T \{ \phi(x) \phi(y) \} | \Omega \rangle$$

Here, $|\Omega\rangle$ is the vacuum state in the interacting theory, (to distinguish it from the vacuum of the free theory, $|0\rangle$)

T represents time-ordering, as before.

This correlation function schematically represents the amplitude of a particle to propagate from y to x .

In the free theory, it's given by the Feynman propagator:

$$\begin{aligned} \langle 0 | T \{ \phi(x) \phi(y) \} | 0 \rangle \\ = D_F(x-y) = \int \frac{d^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip \cdot (x-y)} \end{aligned}$$

We want to see how this changes in the interacting theory. The Hamiltonian of ϕ^4 theory is:

$$H = H_0 + H_{\text{int}} = H_0 + \int d^3x \frac{\lambda}{4!} \phi^4(\vec{x})$$

Here, $\phi(\vec{x})$ is a Schrödinger picture field, while $\phi(x)$ in the correlator is a Heisenberg field.

We want to compute the full correlator as a power series in λ . The interacting part of the theory appears in two places: first, in the definition of the Heisenberg field:

$$\phi(x) = e^{iHt} \phi(\vec{x}) e^{-iHt},$$

and second, in the definition of $|Q\rangle$.

We'll relate both these quantities to free field operators and the free theory vacuum, $|0\rangle$, which we know how to work with.

Let's start with $\phi(x)$. At any fixed time t_0 , we can express ϕ as:

$$\phi(t_0, \vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} + a_{\vec{p}}^\dagger e^{-i\vec{p}\cdot\vec{x}} \right)$$

Then, to obtain $\phi(t, \vec{x})$ for $t \neq t_0$, we just move to the Heisenberg picture:

$$\phi(t, \vec{x}) = e^{iH(t-t_0)} \phi(t_0, \vec{x}) e^{-iH(t-t_0)}$$

For $\lambda = 0$, $H = H_0$ and this reduces to:

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$$\left. \phi(t, \vec{x}) \right|_{\lambda=0} = e^{iH_0(t-t_0)} \phi(t_0, \vec{x}) e^{-iH_0(t-t_0)} \\ = \phi_I(t, \vec{x})$$

where we have defined the interaction picture field,
 $\phi_I(t, \vec{x}) = \phi_I(x)$.

This operator is particularly convenient because we know how to construct it explicitly:

$$\phi_I(t, \vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(a_{\vec{p}} e^{-ip \cdot x} + a_{\vec{p}}^\dagger e^{ip \cdot x} \right)_{x^0=t-t_0}$$

Now we can express the full Heisenberg field in terms of ϕ_I :

$$\phi(t, \vec{x}) = e^{iH(t-t_0)} \underbrace{\left[e^{-iH_0(t-t_0)} \phi_I(t, \vec{x}) e^{+iH_0(t-t_0)} \right]}_{\text{the Schrödinger field}} e^{-iH(t-t_0)}$$

$$= U^\dagger(t, t_0) \phi_I(t, \vec{x}) U(t, t_0)$$

where we've defined the unitary operator

$$U(t, t_0) = e^{iH_0(t-t_0)} e^{-iH(t-t_0)}$$

This is the time-evolution operator in the interaction picture.

Now we want to write $U(t, t_0)$ in terms of ϕ_I , since we have the latter already expressed in terms of ladder operators.

To this end, note that $U(t, t_0)$ solves the Schrödinger equation, and is in fact the unique solution w/ initial condition $U(t_0, t_0) = 1$.

$$i \frac{\partial}{\partial t} U(t, t_0) = i \frac{\partial}{\partial t} \left[e^{iH_0(t-t_0)} e^{-iH(t-t_0)} \right]$$

for an operator, $\frac{d}{dt} e^{At} = A e^{At} = e^{At} A$ (A commutes w/ itself)

$$\begin{aligned} \Rightarrow &= e^{iH_0(t-t_0)} (H - H_0) e^{-iH(t-t_0)} \\ &= e^{iH_0(t-t_0)} H_{\text{int}} e^{-iH(t-t_0)} \\ &= e^{iH_0(t-t_0)} H_{\text{int}} e^{-iH_0(t-t_0)} \underbrace{e^{iH_0(t-t_0)} e^{-iH(t-t_0)}}_{U(t, t_0)} \\ &\equiv H_I(t) U(t, t_0) \end{aligned}$$

where we've defined the perturbation in the interacting picture:

$$\begin{aligned} H_I(t) &\equiv e^{iH_0(t-t_0)} H_{\text{int}} e^{-iH_0(t-t_0)} \\ &= e^{iH_0(t-t_0)} \int d^3x \frac{\lambda}{4!} \phi^4(t_0, \vec{x}) e^{-iH_0(t-t_0)} \\ &= \int d^3x \frac{\lambda}{4!} \phi_I^4(x) \end{aligned}$$

In the last step, we inserted several factors of $e^{-iH_0(t-t_0)} e^{iH_0(t-t_0)} = 1$ b/w the Schrödinger picture fields to get to $\phi_I(x)$.

Now we can solve the differential equation as a series in the small parameter λ . The solution is:

$$U(t, t_0) = 1 + (-i) \int_{t_0}^t dt_1 H_I(t_1) + (-i)^2 \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 H_I(t_1) H_I(t_2) + (-i)^3 \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \int_{t_0}^{t_2} dt_3 H_I(t_1) H_I(t_2) H_I(t_3) + \dots$$

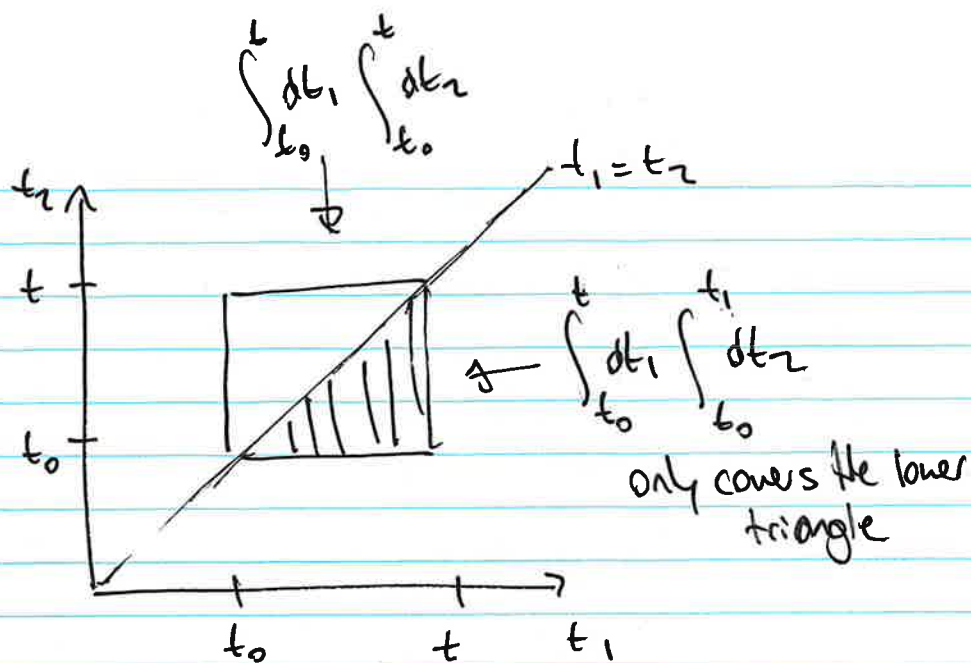
To check this is the solution, just compute $i \frac{\partial}{\partial t} U(t, t_0)$. Each term gives the previous one, times $H_I(t)$. Factor out $H_I(t)$ in front to get $H_I U(t, t_0)$. The initial condition $U(t_0, t_0) = 1$ is clearly satisfied (all integrals vanish).

Note that each term in $U(t, t_0)$ has the factors of $H_I(t)$ arranged in time order — later on the left ($t_1 > t_2 > t_3 > \dots$)

Thus, for instance, we can write

$$\int_{t_0}^{t_1} dt_1 \int_{t_0}^{t_1} dt_2 H_I(t_1) H_I(t_2) = \frac{1}{2} \int_{t_0}^t dt_1 \int_{t_0}^t dt_2 T[H_I(t_1) H_I(t_2)] \quad \text{now } t, \text{ not } t_1!$$

this works b/c the integrand on the RHS is symmetric about $t_2 = t_1$ — the integral on the LHS only covers the lower triangle in the following figure:



This generalizes to higher orders:

$$\begin{aligned}
 & \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \dots \int_{t_0}^{t_{n-1}} dt_n H_I(t_1) \dots H_I(t_n) \\
 &= \frac{1}{n!} \int_{t_0}^t dt_1 \dots \int_{t_0}^t dt_n T\{H_I(t_1) \dots H_I(t_n)\}
 \end{aligned}$$

Thus we can write:

$$\begin{aligned}
 U(t, t_0) &= 1 + (-i) \int_{t_0}^t dt_1 H_I(t_1) + \frac{(-i)^2}{2} \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 T(H_I(t_1) H_I(t_2)) \\
 &\quad + \dots \\
 &= T\left\{ \exp\left[-i \int_{t_0}^t dt' H_I(t')\right] \right\}
 \end{aligned}$$

This is known as the Dyson series.

To recap, we have shown how to express the Heisenberg fields (which appear in the correlator of interest) in terms of the interaction field, $\phi_I(x)$:

$$\phi_I(x) = U^\dagger(t, t_0) \phi_I(x) U(t, t_0)$$

w/ $U(t, t_0)$ expressed in terms of ϕ_I via $H_I(t)$.

Before proceeding, note several useful properties of $U(t, t')$ (where t' is now arbitrary, rather than the time we fixed on Schrödinger fields):

$$U(t, t') = T \exp \left[-i \int_{t'}^t dt'' H_I(t'') \right] \quad (t > t')$$

$$i \frac{\partial}{\partial t} U(t, t') = H_I(t) U(t, t') ; \quad U(t, t) = 1$$

and

$$U(t, t') = e^{iH_0(t-t_0)} e^{-iH(t-t')} e^{-iH_0(t'-t_0)}$$

Note that this implies that U is unitary: $U^\dagger = U^{-1}$.

Finally, U satisfies:

$$U(t_1, t_2) U(t_2, t_3) = U(t_1, t_3)$$

$$U(t_1, t_3) (U(t_2, t_3))^\dagger = U(t_1, t_2)$$

Now we turn to the vacuum state $|\Omega\rangle$.

We want to relate this to the free vacuum state $|0\rangle$ that we know how to compute with. Consider the action of e^{-iHT} on $|0\rangle$, inserting a complete set of energy eigenstates:

$$e^{-iHT}|0\rangle = e^{-iHT} \sum_n |n\rangle \langle n|0\rangle = \sum_n e^{-iE_n T} |n\rangle \langle n|0\rangle$$

w/ E_n the energy eigenvalues of H . (This is an enormously degenerate, continuous sum, but that doesn't matter for our purposes)

We must assume - reasonably - that $\langle \Omega|0\rangle \neq 0$ (otherwise H_{int} could hardly be a "small" perturbation)

Then, the series above contains $|\Omega\rangle$!

$$e^{-iHT}|0\rangle = e^{-iE_0 T} |\Omega\rangle \langle \Omega|0\rangle + \sum_{n \neq 0} e^{-iE_n T} |n\rangle \langle n|0\rangle$$

w/ $E_0 = \langle \Omega|H|\Omega\rangle$ is the ground state energy.

We shift the zero of energy s.t. $H_0|0\rangle = 0$. Then, since $E_n > E_0$, we can remove the terms w/ $n \neq 0$ in the series by taking the large T limit with a small imaginary part:

$$T \rightarrow \infty(1 - i\epsilon)$$

Then all the exponentials have a small damping component, but this is smallest for $n=0$ ($e^{-\epsilon E_n T}$)

So we have

$$|2\rangle = \lim_{T \rightarrow \infty(1-i\epsilon)} \left(e^{-iE_0 T} \langle 2|0\rangle \right)^{-1} e^{-iHT} |0\rangle$$

Now for large T , we can shift $T \rightarrow T+t_0$ so that

$$|2\rangle = \lim_{T \rightarrow \infty(1-i\epsilon)} \left(e^{-iE_0(T+t_0)} \langle 2|0\rangle \right)^{-1} e^{-iH(T+t_0)} |0\rangle$$

insert $1 = e^{-iH_0(-T-t_0)}$

then, looking back at our expression for $U(t, t')$ we recognize

$$e^{iH(-T-t_0)} e^{-iH_0(-T-t_0)} = U^\dagger(-T, t_0) = U(t_0, -T)$$

So

$$|2\rangle = \lim_{T \rightarrow \infty(1-i\epsilon)} \left(e^{-iE_0(t_0 - (-T))} \langle 2|0\rangle \right)^{-1} U(t_0, -T) |0\rangle$$

this is some constant prefactor,

otherwise, we see we obtain $|2\rangle$ by evolving $|0\rangle$ from time $-T$ to t_0 w/ $U(t_0, -T)$. A similar calculation gives:

$$\langle 2| = \lim_{T \rightarrow \infty(1-i\epsilon)} \langle 0| U(T, t_0) \left(e^{-iE_0(T-t_0)} \langle 0|2\rangle \right)^{-1}$$

Now we ~~are~~ have all the pieces we need to compute the correlation function.

Let's assume $x^0 > y^0$. Then,

$$\langle \Omega | \phi(x) \phi(y) | \Omega \rangle = \lim_{T \rightarrow \infty(1-i\epsilon)} \left(e^{-iE_0(T-t_0)} \langle 0 | \Omega \rangle \right)^{-1}$$

$$\times \langle 0 | U(T, t_0) \left\{ [U(x^0, t_0)]^\dagger \phi_I(x) U(x^0, t_0) \right\}$$

$$\times \left\{ [U(y^0, t_0)]^\dagger \phi_I(y) U(y^0, t_0) \right\} U(t_0, -T) | 0 \rangle$$

$$\times \left(e^{-iE_0(t_0 - (-T))} \langle 0 | \Omega \rangle \right)^{-1}$$

$$= \lim_{T \rightarrow \infty(1-i\epsilon)} \left(|\langle 0 | \Omega \rangle|^2 e^{-iE_0(2T)} \right)^{-1}$$

$$\times \langle 0 | U(T, x^0) \phi_I(x^0) U(x^0, y^0) \phi_I(y^0) U(y^0, -T) | 0 \rangle$$

Now we divide this by 1 in the form

$$1 = \langle \Omega | \Omega \rangle$$

$$= \lim_{T \rightarrow \infty(1-i\epsilon)} \left(|\langle 0 | \Omega \rangle|^2 e^{-iE_0(2T)} \right)^{-1} \underbrace{\langle 0 | U(T, t_0) U(t_0, -T) | \Omega \rangle}_{= \langle 0 | U(T, -T) | 0 \rangle}$$

Putting it all together:

$$\langle \Omega | \phi(x) \phi(y) | \Omega \rangle =$$

$$\lim_{T \rightarrow \infty (1-i\epsilon)} \frac{\langle 0 | U(T, x^0) \phi_I(x) U(x^0, y^0) \phi_I(y) U(y^0, -T) | 0 \rangle}{\langle 0 | U(T, -T) | 0 \rangle}$$

Note that all the operators are time ordered. This would've happened if $y^0 > x^0$ too, so we can rewrite this w/ the time ordering symbol on each side:

$$\langle \Omega | T \{ \phi(x) \phi(y) \} | \Omega \rangle$$

$$= \lim_{T \rightarrow \infty (1-i\epsilon)} \frac{\langle 0 | T \{ \phi_I(x) \phi_I(y) \exp \left[-i \int_{-T}^T dt H_I(t) \right] \} | 0 \rangle}{\langle 0 | T \{ \exp \left[-i \int_{-T}^T dt H_I(t) \right] \} | 0 \rangle}$$

A similar formula holds for more factors of ϕ put into the correlator.

This is a big step in what we'll eventually need for scattering amplitude calculations.

~~Next, we'll need to learn how~~

For practical calculations, we keep only the first few terms in the expansion of the exponent.

Now we'll see how to handle the resulting terms efficiently.

Wick's Theorem

To make further progress, we need to evaluate expressions of the form:

$$\langle 0 | T \{ \phi_I(x_1) \phi_I(x_2) \dots \phi_I(x_n) \} | 0 \rangle$$

— time-ordered vacuum expectation values of (interaction picture) free field operators.

For $n=2$, this is the Feynman propagator:

$$\langle 0 | T \{ \phi_I(x_1) \phi_I(x_2) \} | 0 \rangle = D_F(x_1 - x_2)$$

* From here on, I'll drop the subscript I , but remember we're always dealing w/ interaction picture field operators. *

Start w/ the case of two fields,

$$\langle 0 | T \{ \phi(x) \phi(y) \} | 0 \rangle$$

To help generalize this, split ϕ into positive & negative frequency parts:

$$\phi(x) = \phi^+(x) + \phi^-(x)$$

$$\text{w/ } \phi^+(x) = \int \frac{d^3p}{(2\pi)^4} \frac{1}{\sqrt{2E_{\vec{p}}}} a_{\vec{p}} e^{ip \cdot x}$$

$$\phi^-(x) = \int \frac{d^3p}{(2\pi)^4} \frac{1}{\sqrt{2E_{\vec{p}}}} a_{\vec{p}}^\dagger e^{ip \cdot x}$$

This is useful because

$$\phi^+(x)|0\rangle = 0, \quad \langle 0|\phi^-(x) = 0$$

Now, assume ~~that~~ first that $x^0 > y^0$. Then,

$$\begin{aligned} T\{\phi(x)\phi(y)\}_{x^0 > y^0} &= \phi^+(x)\phi^+(y) + \phi^+(x)\phi^-(y) \\ &\quad + \phi^-(x)\phi^+(y) + \phi^-(x)\phi^-(y) \\ &= \phi^+(x)\phi^+(y) + \phi^-(y)\phi^+(x) + \phi^-(x)\phi^+(y) + \phi^-(x)\phi^-(y) \\ &\quad + [\phi^+(x)\phi^-(y)] \end{aligned}$$

In every term except the commutator, the a 's are positioned to the right of the $a^\dagger$'s. Such terms, such as $(a_p^\dagger a_q^\dagger a_k a_l)$ are said to be in normal order, and have vanishing expectation value.

We define the Normal Ordering symbol $N(\)$ which ~~means~~ puts the operators in normal order.

$$N(a_p a_k^\dagger a_q) = a_k^\dagger a_p a_q$$

ordering irrelevant since they commute.

(sometimes $N(\dots)$ is denoted by $:\dots:$)

Next, consider the case $y^0 > x^0$. Then,

$$T\{\phi(x)\phi(y)\} \underset{y^0 > x^0}{=} \phi^\dagger(y)\phi^\dagger(x) + \phi^\dagger(y)\phi^-(x) \\ + \phi^-(y)\phi^\dagger(x) + \phi^-(y)\phi^-(x)$$

all normal ordered

$$= \phi^\dagger(y)\phi^\dagger(x) + \phi^-(x)\phi^\dagger(y) + \phi^-(y)\phi^\dagger(x) + \phi^-(y)\phi^-(x) \\ + [\phi^\dagger(y), \phi^-(x)]$$

Now we define the contraction of two fields as

$$\overline{\phi(x)\phi(y)} = \begin{cases} [\phi^\dagger(x), \phi^-(y)] & \text{for } x^0 > y^0 \\ [\phi^\dagger(y), \phi^-(x)] & y^0 > x^0 \end{cases}$$

This is in fact, the Feynman propagator!

$$\overline{\phi(x)\phi(y)} = D_F(x-y) = \int \frac{d^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip \cdot (x-y)}$$

as we can check case by case! for $x^0 > y^0$:

$$[\phi^\dagger(x), \phi^-(y)] = \int \frac{d^3 p}{(2\pi)^3} \int \frac{d^3 q}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \frac{1}{\sqrt{2E_q}} e^{-ip \cdot x + iq \cdot y} \underbrace{[a_p, a_q^\dagger]}_{(2\pi)^3 \delta^3(\vec{p} - \vec{q})} \\ = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip \cdot (x-y)} \Big|_{p^0 = E_p} = D_F(x-y)$$

The same calculation w/ $y^0 > x^0$ gives us the opposite phase, as desired.

To summarize, we have

$$T\{\phi(x)\phi(y)\} = N\{\phi(x)\phi(y) + \overline{\phi(x)\phi(y)}\}$$

This generalizes (by induction) to products of more fields. The result is Wick's Theorem:

$$T\{\phi(x_1)\phi(x_2)\dots\phi(x_n)\} = N\{\phi(x_1)\phi(x_2)\dots\phi(x_n) + (\text{all possible contractions})\}$$

where all possible contractions means every way of pairing the fields. E.g., for $n=4$, writing $\phi(x_n) = \phi_n$:

$$\begin{aligned} T\{\phi_1\phi_2\phi_3\phi_4\} = & N\{\phi_1\phi_2\phi_3\phi_4 \\ & + \overline{\phi_1\phi_2}\phi_3\phi_4 + \overline{\phi_1\phi_3}\phi_2\phi_4 + \overline{\phi_1\phi_4}\phi_2\phi_3 \\ & + \phi_1\overline{\phi_2\phi_3}\phi_4 + \phi_1\overline{\phi_2\phi_4}\phi_3 + \phi_1\phi_2\overline{\phi_3\phi_4} \\ & + \overline{\phi_1\phi_2}\overline{\phi_3\phi_4} + \overline{\phi_1\phi_3}\overline{\phi_2\phi_4} + \overline{\phi_1\phi_4}\overline{\phi_2\phi_3}\} \end{aligned}$$

And the convention is that two non-adjacent operators contracted still results in a Feynman propagator:

$$N(\overline{\phi_1 \phi_2} \phi_3 \phi_4) = D_F(x_1 - x_3) N(\phi_2 \phi_4)$$

Now, taking the vacuum expectation value, any normal ordered product vanishes: $\langle 0 | N(\phi_1 \phi_2 \dots) | 0 \rangle = 0!$

→ Only terms which are fully contracted give a nonzero result!

⇒ for four fields:

$$\langle 0 | T \{ \phi_1 \phi_2 \phi_3 \phi_4 \} | 0 \rangle$$

$$= \langle 0 | \overline{\phi_1 \phi_2} \overline{\phi_3 \phi_4} + \overline{\phi_1 \phi_2} \phi_3 \phi_4 + \overline{\phi_1 \phi_2} \phi_3 \phi_4 | 0 \rangle$$

$$= D_F(x_1 - x_2) D_F(x_3 - x_4) + D_F(x_1 - x_3) D_F(x_2 - x_4)$$

$$+ D_F(x_1 - x_4) D_F(x_2 - x_3)$$

⊗ We now know how to simplify terms in the expansion. It's time to make our lives a lot simpler by introducing Feynman diagrams.

Brief recap:

We've shown that, to compute time-ordered correlation functions of fields, we compute:

$$\begin{aligned} & \langle \Omega | T \{ \phi(x_1) \phi(x_2) \dots \} | \Omega \rangle \\ &= \lim_{T \rightarrow \infty (1-i\epsilon)} \frac{\langle 0 | T \{ \phi_I(x_1) \phi_I(x_2) \dots \exp \left[-i \int_{-T}^T dt H_I(t) \right] \} | 0 \rangle}{\langle 0 | T \left\{ \exp \left(i \int_{-T}^T dt H_I(t) \right) \right\} | 0 \rangle} \end{aligned}$$

If H_I contains a small parameter, we can expand this exponential. We then get time-ordered free-vacuum expectation values of interaction-picture fields:

$$\langle 0 | T \{ \phi_I(x_1) \phi_I(x_2) \dots \phi_I(x_n) \} | 0 \rangle$$

For $n=2$ (dropping the I subscript from here on):

$$\langle 0 | T \{ \phi(x_1) \phi(x_2) \} | 0 \rangle = D_F(x_1 - x_2)$$

↑
the Feynman propagator

For larger n , the result is obtained by Wick's Theorem:

$$\begin{aligned} & T\{\phi(x_1)\phi(x_2)\dots\phi(x_n)\} \\ &= N\{\phi(x_1)\phi(x_2)\dots\phi(x_n)\} \\ &\quad + (\text{all possible contractions}) \end{aligned}$$

where N = normal ordering, w/ all the a^+ 's to the left of the a 's.

This simplifies in any correlator into a sum of products of Feynman propagators.

Now let's make our lives simpler using Feynman diagrams!

Feynman Diagrams

To simplify our lives, let's give a diagrammatic expression of the 4-point correlator we found last time.

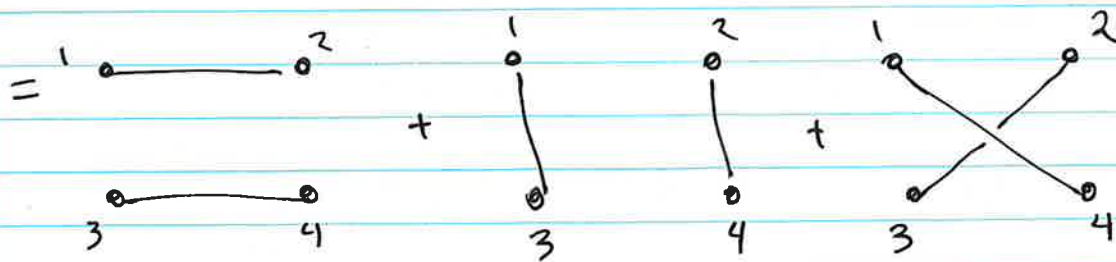
$$\langle 0 | T \{ \phi_1 \phi_2 \phi_3 \phi_4 \} | 0 \rangle$$

$$= D_F(x_1 - x_2) D_F(x_3 - x_4) + D_F(x_1 - x_3) D_F(x_2 - x_4) + D_F(x_1 - x_4) D_F(x_2 - x_3)$$

We'll associate w/ each point x_i a dot, and a factor $D_F(x_i - x_j)$ as a line connecting the two dots.

Then,

$$\langle 0 | T \{ \phi_1 \phi_2 \phi_3 \phi_4 \} | 0 \rangle$$



Physically, we can think of particles being created at two points, propagating to one of the others, and then being annihilated.

The three diagrams enumerate the ways this can happen, and the correlation function is the sum of these.

For our actual calculations of interest, we have factors of

$$H_I = \int d^3x \frac{\lambda}{4!} \phi^4$$

→ all fields at the same point!

Consider the two-point function:

$$\langle 0 | T \{ \phi(x) \phi(y) \} | 0 \rangle$$

$$= \lim_{T \rightarrow \infty (1-i\epsilon)} \frac{\langle 0 | T \{ \phi_I(x) \phi_I(y) \exp \left[-i \int_{-T}^T dt H_I(t) \right] \} | 0 \rangle}{\langle 0 | T \{ \exp \left[-i \int_{-T}^T dt H_I(t) \right] \} | 0 \rangle}$$

Focus on the numerator for now — to first order, this is given by

$$\langle 0 | T \left\{ \phi(x) \phi(y) + \phi(x) \phi(y) \left[-i \int_{-T}^T dt H_I(t) \right] + \dots \right\} | 0 \rangle$$

The first term is the free theory result:

$$\langle 0 | T \{ \phi(x) \phi(y) \} | 0 \rangle = D_F(x-y) = \text{diagram of a line connecting } x \text{ and } y$$

For ϕ^4 theory, the second term is

$$\begin{aligned} & \langle 0 | T \{ \phi(x) \phi(y) (-i) \int d^4z \frac{\lambda}{4!} \phi^4(z) \} | 0 \rangle \\ &= \left(\frac{-i\lambda}{4!} \right) \langle 0 | T \{ \phi(x) \phi(y) \int d^4z \phi^4(z) \} | 0 \rangle \end{aligned}$$

This is a (large) time-ordered correlator. we can apply Wick's theorem, and drop the normal ordered pieces, and any w/ uncontracted fields, since these vanish in the expectation value.

Then we get one term for every way of contracting the six fields in pairs.

There are two classes of ways to do this:

- 1) $\phi(x)$ & $\phi(y)$ are contracted to each other, while the 4 fields in $\phi^4(z)$ are contracted to each other!

$$\begin{aligned} & \overbrace{\phi(x) \phi(y)} \int d^4z \overbrace{\phi(z) \phi(z)} \overbrace{\phi(z) \phi(z)} \\ &+ \overbrace{\phi(x) \phi(y)} \int d^4z \overbrace{\phi(z) \phi(z)} \overbrace{\phi(z) \phi(z)} \\ &+ \phi(x) \phi(y) \int d^4z \overbrace{\phi(z) \phi(z)} \overbrace{\phi(z) \phi(z)} \end{aligned}$$

these are all equal to

$$D_F(x-y) \int d^4z D_F(z-x) D_F(z-y)$$

2) $\phi(x)$, $\phi(y)$ are each contracted with $\phi(z)$ fields, w/ one pair of $\phi(z)$ fields contracted. One such term:

$$\overbrace{\phi(x) \phi(y)} \int d^4z \overbrace{\phi(z) \phi(z)} \overbrace{\phi(z) \phi(z)}$$

Same counting carefully gives 12 terms like this, all equal to

$$\int d^4z D_F(x-z) D_F(y-z) D_F(z-z)$$

So,

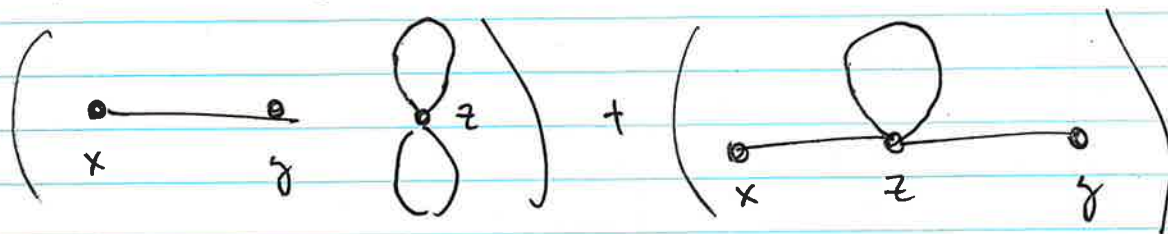
$$\langle 0 | T \{ \phi(x) \phi(y) (-i) \int d^4z \frac{\lambda}{4!} \phi^4(z) \} | 0 \rangle$$

$$= 3 \left(-\frac{i\lambda}{4!} \right) D_F(x-y) \int d^4z D_F(z-x) D_F(z-y)$$

$$+ 12 \left(-\frac{i\lambda}{4!} \right) \int d^4z D_F(x-z) D_F(y-z) D_F(z-z)$$

This expression can be represented diagrammatically by amending our earlier rules w/ an internal point, w/ 4 lines connected to it, $\sim -i\lambda \int d^4z$.

We get two diagrams:



The lines are referred to as propagators, as they are associated w/ a factor D_F .

Internal points where four lines meet are known as vertices.

The first term has the ordinary propagator (the same as the first term in the expansion) times a "disconnected" piece. We'll revisit this shortly.

The second is a correction to the ordinary propagator, with a loop, which appears at first order in the coupling constant.

Let's do one more complicated example! The λ^3 term in the expansion of the correlator contains a piece:

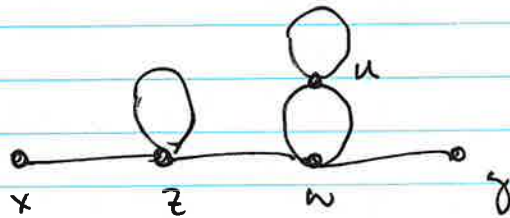
$$\begin{aligned} & \langle 0 | \phi(x) \phi(y) \frac{1}{3!} \left(\frac{-i\lambda}{4!} \right)^3 \int d^4z \phi^4(z) \int d^4w \phi^4(w) \int d^4u \phi^4(u) | 0 \rangle \\ &= \frac{1}{3!} \left(\frac{-i\lambda}{4!} \right)^3 \int d^4z d^4w d^4u D_F(x-z) D_F(z-z) D_F(z-w) D_F(w-y) \\ & \quad \times D_F(w-u) D_F(w-u) D_F(u-u) \end{aligned}$$

There are lots of different contractions that give the same expression. We can count them.

$$\underbrace{3!}_{\text{interchange of vertices}} \times \underbrace{4 \cdot 3}_{\text{placement into } z \text{ vertex}} \times \underbrace{4 \cdot 3 \cdot 2}_{\text{placement into } w \text{ vertex}} \times \underbrace{4 \cdot 3}_{\text{placement into } u \text{ vertex}} \times \underbrace{1/2}_{\text{interchange of } w-u \text{ contractions}}$$

The product of all these factors gives 10,368 contractions leading to the same expression.

We represent this by a single diagram.



How do we determine the overall constant for a general amplitude? There is a factor of $\int d^4x (-i\lambda/4!)$ associated w/ each vertex. There is another factor of $1/n!$ from the Dyson series associated w/ n powers of $H_{\pm}$, but this usually cancels the $n!$ from interchanging vertices.

There is also frequently a factor of $4!$ from the number of ways to contract 4 lines w/ four fields in each vertex, which cancels the denominator in $-i\lambda/4!$. — This is why that normalization for λ is so useful.

Thus, by convention, we associate a factor $\int d^4z (-i\lambda)$ for each vertex.

For some diagrams (such as the last one!) this counting scheme leads to a constant that is too large by a symmetry factor S .

We see this for our 2^3 term, w/ $S = 2 \cdot 2 \cdot 2 = 8$.

Two factors of 2 come from lines starting & ending at the same vertex:

$$\int d^4z \underbrace{\phi \phi \phi \phi}_{\text{symmetric under interchange of the ends of the line.}} \longleftrightarrow \text{diagram of a loop attached to a vertex labeled } z$$

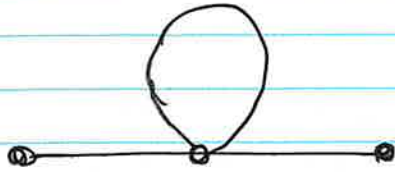
The other factor of 2 came from the two propagators connecting w to u .

$$\int d^4w \underbrace{\phi \phi \phi \phi}_{\text{these two}} \int d^4u \underbrace{\phi \phi \phi \phi}_{\text{these two}} \rightarrow \text{diagram of two loops sharing a vertex, with external lines labeled } w \text{ and } u$$

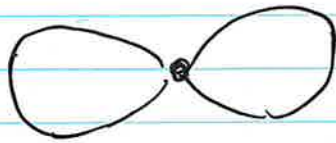
A third possible symmetry factor is the equivalence of two vertices.

To obtain the correct constant for a diagram, we divide by the factor S .

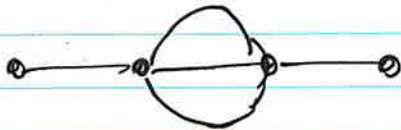
We won't generally care about complicated diagrams in this course, but for completeness, some examples are given below.



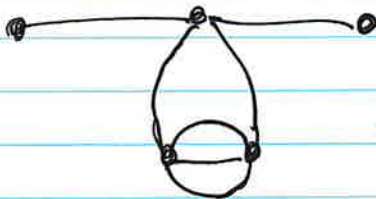
$$S = 2$$



$$S = 2 \cdot 2 \cdot 2 = 8$$



$$S = 3! = 6$$



$$S = 3! \cdot 2 = 12$$

In practice, it is sometimes easier to just count the number of contractions, like we did for the λ^3 example.

Putting this all together, we get the Feynman rules for computing a correlation function.

$$\langle 0 | T \{ \phi_I(x_1) \phi_I(x_2) \dots \phi_I(x_n) \exp \left[-i \int dt H_I(t) \right] \} | 0 \rangle$$

$$= \left(\text{sum of all possible diagrams w/ } n \text{ external points} \right)$$

where we associate an analytic expression w/ each diagram as follows.

1. For each propagator,  $= D_F(x-y)$

2. For each vertex,  $= -i\lambda \int d^4z$

3. For each external point:  $= 1$

4. Divide by the symmetry factor.

Physically, we associate $-i\lambda$ w/ the probability to create/annihilate particles at a vertex. The integral sums ~~at~~ over all the spacetime points where this can happen.

This is a manifestation of the superposition principle in QM. — we have to add all the amplitudes for a given process.

The analytic result for a diagram is the product of all the factors from steps 1-4 above. The full correlator is the sum of each diagram's total factor.

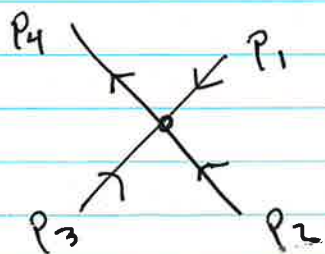
The above rules are written in terms of spacetime points, x & y — they are sometimes called the position space Feynman rules.

Practically speaking, it will often be more useful to work in momentum space. This follows from the Fourier expansion:

$$D_F(x-y) = \int \frac{d^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip \cdot (x-y)}$$

Diagrammatically, this is represented by assigning a momentum to each propagator, indicating the direction w/ an arrow. Since $D_F(x-y) = D_F(y-x)$, this direction is arbitrary.

At a vertex, the z -dependent factors are:

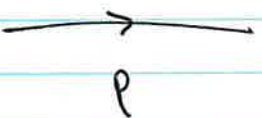


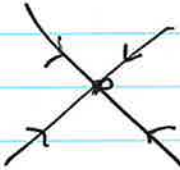
$$\begin{aligned} &\longleftrightarrow \int d^4 z e^{-ip_1 \cdot z} e^{-ip_2 \cdot z} e^{-ip_3 \cdot z} e^{ip_4 \cdot z} \\ &= (2\pi)^4 \delta^4(p_1 + p_2 + p_3 - p_4) \end{aligned}$$

→ 4-momentum is conserved at each vertex.

These δ -functions are used to carry out some of the momentum integrals.

We can work out the equivalent position space Feynman rules for ϕ^4 theory:

1. For each propagator,  $= \frac{i}{p^2 - m^2 + i\epsilon}$

2. For each vertex,  $= -i\lambda$

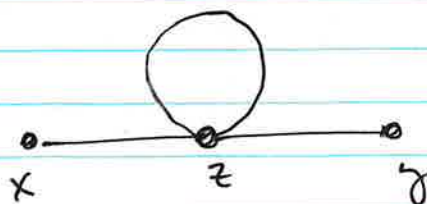
3. For each external point,   $= e^{-ip \cdot x}$

4. Impose momentum conservation at each vertex.

5. Integrate over undetermined momentum.

6. Divide by the symmetry factor.

As an example, return to the $\mathcal{O}(2)$ correction to the two-point correlator, given by



The expression in position space was:

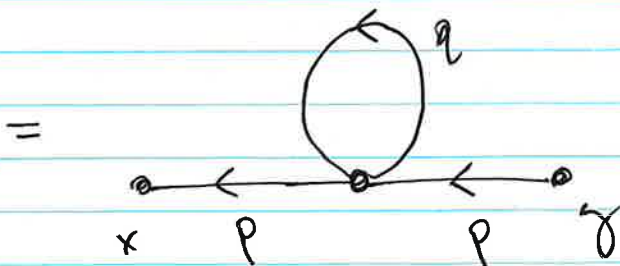
$$= \frac{1}{2} (-i\lambda) \int d^4 z D_F(x-z) D_F(y-z) D_F(z-z)$$

Plugging in the Fourier expansion:

$$= \frac{1}{2} (-i\lambda) \int d^4 z \left[\int \frac{d^4 p}{(2\pi)^4} \frac{i e^{-ip(x-z)}}{p^2 - m^2 + i\epsilon} \right] \times \left[\int \frac{d^4 k}{(2\pi)^4} \frac{i e^{-ik(y-z)}}{k^2 - m^2 + i\epsilon} \right] \\ \times \left[\int \frac{d^4 q}{(2\pi)^4} \frac{i e^{-iq(z-z)}}{q^2 - m^2 + i\epsilon} \right]$$

$$= \frac{1}{2} (-i\lambda) \int \frac{d^4 q}{(2\pi)^4} \frac{i}{q^2 - m^2 + i\epsilon} \int \frac{d^4 p}{(2\pi)^4} \frac{i e^{-ip \cdot x}}{p^2 - m^2 + i\epsilon} \int \frac{d^4 k}{(2\pi)^4} \frac{i e^{-ik \cdot y}}{k^2 - m^2 + i\epsilon} \\ \times \underbrace{\int d^4 z e^{i(p+k) \cdot z}}_{= (2\pi)^4 \delta^4(p+k)}$$

$$= \frac{1}{2} (-i\lambda) \int \frac{d^4 q}{(2\pi)^4} \frac{i}{q^2 - m^2 + i\epsilon} \int \frac{d^4 p}{(2\pi)^4} \frac{i}{(p^2 - m^2 + i\epsilon)} \frac{i}{(p^2 - m^2 + i\epsilon)} e^{-ip(x-y)}$$

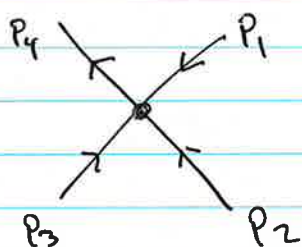


Now let's return to both the large T limit, and the denominator of our correlation function:

$$\langle \Omega | T \{ \phi(x_1) \dots \phi(x_n) \} | \Omega \rangle$$

$$= \lim_{T \rightarrow \infty(1-i\epsilon)} \frac{\langle 0 | T \{ \phi_I(x_1) \dots \phi_I(x_n) \exp[-i \int_{-T}^T dt H_I(t)] \} | 0 \rangle}{\langle 0 | T \{ \exp[-i \int_{-T}^T dt H_I(t)] \} | 0 \rangle}$$

We neglected the large T dependence in all our calculations before. Technically, our position space vertex should have been:



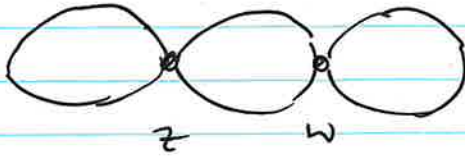
$$\longleftrightarrow \lim_{T \rightarrow \infty(1-i\epsilon)} \int_{-T}^T dz^0 \int d^3 z e^{-i(p_1 + p_2 + p_3 - p_4) \cdot z}$$

The $-i(p_1^0 + p_2^0 + p_3^0 - p_4^0) \cdot z^0$ part of the exponent blows up unless the argument is purely imaginary. This is problematic since we have an $i\epsilon T$ part leading to a real part. To regulate this, we shift $p^0 \rightarrow p^0(1+i\epsilon)$

→ This is precisely the Feynman prescription for the contour integral in $D_F(x-y)$.

This cancels the problematic part, so we can safely take the $T \rightarrow \infty$ limit.

Consider next a disconnected diagram:



Using the position space Feynman rules, this gives:

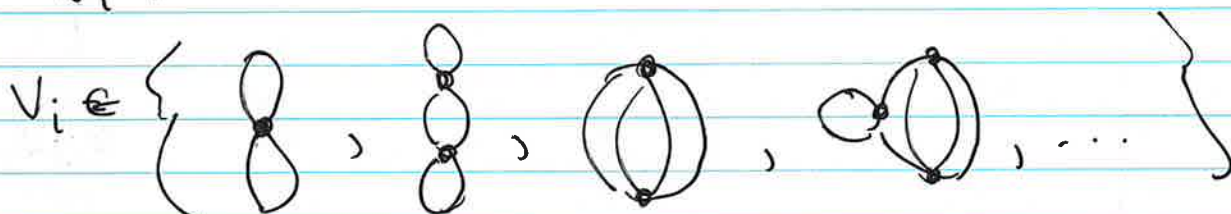
$$\begin{aligned}
 & \sim (-i\lambda)^2 \int d^4z d^4w D_F(z-z) D_F(w-w) D_F(z-w) D_F(z-w) \\
 & = (-i\lambda)^2 \int d^4z \int d^4w \int \frac{d^4p_3}{(2\pi)^4} \frac{i}{p_3^2 - m^2 + i\epsilon} \int \frac{d^4p_4}{(2\pi)^4} \frac{i}{p_4^2 - m^2 + i\epsilon} \\
 & \quad \times \int \frac{d^4p_1}{(2\pi)^4} \frac{ie^{-ip_1 \cdot (z-w)}}{p_1^2 - m^2 + i\epsilon} \int \frac{d^4p_2}{(2\pi)^4} \frac{ie^{-ip_2 \cdot (z-w)}}{p_2^2 - m^2 + i\epsilon} \\
 & = (-i\lambda)^2 \int d^4p_1 \dots d^4p_4 (\dots) \\
 & \quad \times \int d^4w e^{i(p_1+p_2) \cdot w} \underbrace{\int d^4z e^{-i(p_1+p_2) \cdot z}}_{(2\pi)^4 \delta^4(p_1+p_2)} \rightarrow \text{do integral over } p_2 \\
 & = (-i\lambda)^2 \int \frac{d^4p_3}{(2\pi)^4} \int \frac{d^4p_4}{(2\pi)^4} \frac{i}{p_3^2 - m^2 + i\epsilon} \frac{i}{p_4^2 - m^2 + i\epsilon} \\
 & \quad \times \int \frac{d^4p_1}{(2\pi)^4} \frac{i}{p_1^2 - m^2 + i\epsilon} \frac{i}{p_1^2 - m^2 + i\epsilon} \times \underbrace{\int d^4w}_{=(2\pi)^4 \delta^4(0)}
 \end{aligned}$$

The factor $\int d^4w \rightarrow 2T$. (volume of space)

Any disconnected diagram will have a similar factor.

Fortunately, we can reorganize our sum over all the disconnected pieces in a useful way.

A typical contribution to an n -point function will have connected pieces and a set of disconnected pieces we'll label V_i .



A contribution can have n_i pieces of the form V_i , for each i , in addition to the connected diagrams.

For any given piece, only finitely many of the n_i will be nonzero. If we also let V_i denote the value of the piece V_i , the value of the contribution is:

$$\left(\begin{array}{c} \text{value of connected} \\ \text{piece} \end{array} \right) \cdot \prod_i \frac{1}{n_i!} (V_i)^{n_i}$$

symmetry factor,
for the n_i copies of V_i .

The sum of all contributions in the numerator of our formula becomes:

$$\sum_{\text{all possible connected pieces}} \cdot \sum_{\text{all } \{n_i\}} \left(\begin{array}{c} \text{value of connected} \\ \text{piece} \end{array} \right) \cdot \prod_i \frac{1}{n_i!} (V_i)^{n_i}$$

↗ all ordered sets $\{n_1, n_2, n_3, \dots\}$
of non-negative integers.

The sum over the connected pieces factors out, so we find:

$$\begin{aligned} &\rightarrow (\bar{\Sigma} \text{ connected}) \times \sum_{\text{all } \{n_i\}} \left(\prod_i \frac{1}{n_i!} (V_i)^{n_i} \right) \\ &= (\bar{\Sigma} \text{ connected}) \left(\sum_{n_1} \frac{1}{n_1!} V_1^{n_1} \right) \left(\sum_{n_2} \frac{1}{n_2!} V_2^{n_2} \right) (\dots) \\ &= (\bar{\Sigma} \text{ connected}) \prod_i \left(\sum_{n_i} \frac{1}{n_i!} V_i^{n_i} \right) \\ &= (\bar{\Sigma} \text{ connected}) \times \prod_i \exp(V_i) \\ &= (\bar{\Sigma} \text{ connected}) \exp \left(\sum_i V_i \right) \end{aligned}$$

In other words, the numerator is all the connected diagrams times the exponential of the sum of all disconnected diagrams.

Now, consider the denominator. Here there are no external points, so we get a series of diagrams with only internal points.

We can reorganize this series in the same way!

$$\lim_{T \rightarrow \infty} \frac{1}{(1-i\epsilon)} \langle 0 | T \exp \left[-i \int_{-T}^T dt H_I(t) \right] | 0 \rangle$$

$$= \exp \left[\text{diagram 1} + \text{diagram 2} + \dots \right]$$

Diagram 1: A single loop (bubble) diagram.

Diagram 2: Two loops connected by a vertical line (a chain of two bubbles).

→ the exponentials of the sum over disconnected diagrams cancel!

Our final formula for an n -point function in ϕ^4 theory is:

$$\langle 0 | T \{ \phi(x_1) \dots \phi(x_n) \} | 0 \rangle$$

$$= \left(\text{sum of all connected diagrams w/ } n \text{ external points} \right)$$

Note that we do not require the diagrams to be fully connected. Only the disconnected diagrams that fully separate are factored out and exponentiated.

Terms w/ some pieces disconnected from each other still contribute to the correlation function.

For instance,

$$\langle \Omega | T \phi_1 \phi_2 \phi_3 \phi_4 | \Omega \rangle$$

$$= \text{---} + \text{---} + \text{---}$$

$$+ \text{---} + \text{---} + \dots$$

$$+ \text{---} + \text{---} + \dots + \text{---} + \dots$$

All of these contribute to the correlation function, but there is an important distinction when some external points are disconnected from each other. We'll revisit this later.

Before moving on, it's useful to give a physical interpretation of the ~~vacuum~~ disconnected diagrams. Rearranging our earlier expression for the two-point correlator,

$$\lim_{T \rightarrow \infty (1-i\epsilon)} \langle 0 | T \{ \phi_I(x) \phi_I(y) \exp \left[-i \int_{-T}^T dt H_I(t) \right] \} | 0 \rangle$$

$$= \langle \Omega | \phi(x) \phi(y) | \Omega \rangle \times \lim_{T \rightarrow \infty (1-i\epsilon)} (| \langle 0 | \Omega \rangle |^2 e^{-iE_0/2T})$$

The terms w/ T dependence on each side suggest the identification,

$$\exp\left[\sum V_i\right] \propto \exp\left[-i E_0(2T)\right]$$

Since each disconnected diagram V_i gives a factor of $(2\pi)^4 \delta^4(0) = 2 \cdot T \cdot \text{Vol.}$, this gives us a formula for the energy density of the vacuum, measured relative to the zero of energy set by $H|0\rangle = |0\rangle$.

$$\frac{E_0}{\text{Vol}} = \frac{i \sum V_i}{2T \cdot \text{Vol}} = \frac{\left[\text{loop} + \text{2-loop} + \text{3-loop} + \dots \right]}{(2\pi)^4 \delta^4(0)}$$

These disconnected diagrams thus do contribute to the physics, but not to the problem ~~is~~ represented by the correlators, and this is why they factor out & cancel.

Scattering & the S Matrix

We now know how to compute correlation functions in perturbative QFTs — these are the foundational objects underlying the theory. However, they're not the most interesting or accessible for particle physics.

Instead, we're interested in transitions between non-trivial initial & final states (as opposed to vacuum to vacuum processes).

We'll set up this problem, and see the important role played by the "S-matrix" operator — this is the time evolution operator $U(t, t')$ in the asymptotic limits $t \rightarrow \infty$, $t' \rightarrow -\infty$.

We'll soon see how entries of the S-matrix are directly related to measurable quantities, such as cross-sections or decay rates.

One missing ingredient is the Lehmann-Symanzik-Zimmerman (LSZ) reduction formula, which will tell us how to extract S-matrix elements from the correlation functions we now know how to compute. From here, we'll be able to calculate these matrix elements from Feynman diagrams.

Let us formulate our scattering problem carefully, and then lay out this procedure in detail.

In a scattering process, we assume some (usually one or two) particles approach each other from macroscopic distances, interact in a microscopically small region, and then the product of the reaction (possibly different particles) become macroscopically separated again.

→ We want the amplitude for some set of particles (as $t \rightarrow -\infty$) to transition to another set (as $t \rightarrow +\infty$)

We'll take the incoming, outgoing particles to be n -particle states:

$$|\vec{p}_1, \dots, \vec{p}_n\rangle = \sqrt{2E_{\vec{p}_1}} a_{\vec{p}_1}^+ \dots \sqrt{2E_{\vec{p}_n}} a_{\vec{p}_n}^+ |0\rangle$$

→ eigenstates of H_0 , not $H_0 + H_{int}$, but we'll see that the correct in/out states are closely related.

~~Consider~~

To see how this works, consider two types of states that transform differently under time translation:

$$|\Phi\rangle \mapsto e^{-iH_0 t} |\Phi\rangle \quad \leftarrow \text{free evolution}$$

$$|\Psi\rangle \mapsto e^{-iH t} |\Psi\rangle \quad \leftarrow \text{free + int. evolution (interacting theory)}$$

As we take $t \rightarrow -\infty$, we'll get our "in" states.

But at macroscopic distances, the particles shouldn't be feeling interactions, so

$$\lim_{t \rightarrow -\infty} e^{-iH t} |\Psi\rangle = \lim_{t \rightarrow -\infty} e^{-iH_0 t} |\Phi\rangle$$

Inverting this, we define the correct "in" state as

$$|\Phi\rangle_{in} \equiv \lim_{t \rightarrow -\infty} e^{iHt} e^{-iH_0 t} |\Phi\rangle$$

$$= \lim_{t \rightarrow -\infty} \Omega(t) |\Phi\rangle = \Omega(-\infty) |\Phi\rangle$$

where we've defined the operator

$$\Omega(t) = e^{iHt} e^{-iH_0 t}$$

Both $|\Phi\rangle_{in}$ and $|\Phi\rangle$ can be labelled in terms of same particle momenta:

$$\underbrace{|\vec{p}_A \vec{p}_B \dots\rangle_{in}}_{\text{corresp. multiparticle state of interacting theory}} = \Omega(-\infty) \underbrace{|\vec{p}_A \vec{p}_B \dots\rangle}_{\text{multiparticle state of free theory}}$$

In complete analogy, we can define the "out" states:

$$|\vec{p}_A \vec{p}_B \dots\rangle_{out} = \Omega(+\infty) |\vec{p}_A, \vec{p}_B, \dots\rangle_{(free)}$$

The point here is that the multiparticle states of the free theory are completely well defined, so we adopt this label for the interacting in/out states, which are only different by a (non-trivial) time translation.

Now, we're interested in the probability amplitude for the scattering process:

$$\phi(p_A) + \phi(p_B) \rightarrow \phi(p_1) + \phi(p_2) + \dots$$

→ this is simply the overlap between the "in" and "out" states:

$$P_{if} = \langle \vec{p}_1, \vec{p}_2, \dots | \vec{p}_A, \vec{p}_B, \dots \rangle_{in}$$

$$= \langle \vec{p}_1, \vec{p}_2, \dots | \mathcal{U}^\dagger(+\infty) \mathcal{U}(-\infty) | \vec{p}_A, \vec{p}_B, \dots \rangle$$

$$= \lim_{\substack{t_i \rightarrow -\infty \\ t_f \rightarrow +\infty}} \langle \vec{p}_1, \vec{p}_2, \dots | \underbrace{e^{iH_0 t_f} e^{-iH t_f} e^{iH t_i} e^{-iH_0 t_i}}_{\text{free states}} | \vec{p}_A, \vec{p}_B, \dots \rangle$$

$$\equiv \mathcal{U}(t_f, t_i)$$

$$= \langle \vec{p}_1, \vec{p}_2, \dots | \mathcal{U}(\infty, -\infty) | \vec{p}_A, \vec{p}_B, \dots \rangle$$

This is the same time-evolution operator we defined before:

$$\mathcal{U}(t, t') = e^{iH_0(t-t_0)} e^{-iH(t-t')} e^{-iH_0(t-t_0)}$$

w/ some reference time t_0 (which we set to $t_0 = 0$)

With these conventions, we define the S-matrix operator:

$$S = U(\infty, -\infty)$$

So the probability amplitude is simply the S-matrix element:

$$\begin{aligned} P_{if} = S_{if} &\equiv \langle \vec{p}_1 \vec{p}_2 \dots | \vec{p}_A \vec{p}_B \dots \rangle_{in} \\ &= \langle \vec{p}_1 \vec{p}_2 \dots | S | \vec{p}_A \vec{p}_B \dots \rangle \end{aligned}$$

To restate this: we've defined S as the operator that performs the evolution between asymptotic in/out states, and written this in the basis of free-field multiparticle states, and in this basis, identify the transition amplitude as these matrix elements.

Fortunately, we've already derived a closed-form expression for $U(t, t')$ in terms of interaction picture fields:

$$S = T \left\{ \exp \left[-i \int_{-\infty}^{\infty} dt H_I(t) \right] \right\}$$

$\nearrow$
time ordering

$\nearrow$
the interaction picture Hamiltonian,
written in terms of the ϕ_I

It is common to decompose the S-matrix as:

$$S = \mathbb{1} + iT$$

The $\mathbb{1}$ represents the trivial case where the particles do not interact, while the T-matrix (not to be confused w/ the time-ordering operator) is the non-trivial scattering probability we're interested in.

Clearly, S (and T) matrix elements should ~~reflect~~ respect 4-momentum conservation. Thus, we can write

$$\begin{aligned} \langle \vec{p}_1 \vec{p}_2 \dots | iT | \vec{p}_A \vec{p}_B \dots \rangle \\ = i \mathcal{M}(\vec{p}_A \vec{p}_B \dots \rightarrow \vec{p}_1 \vec{p}_2 \dots) \times (2\pi)^4 \delta^4(\vec{p}_A + \vec{p}_B + \dots - \vec{p}_1 - \vec{p}_2 - \dots) \end{aligned}$$

The quantity $\mathcal{M}$ is known as the invariant matrix element or Feynman amplitude or just "the amplitude".

The Feynman rules will soon give us a prescription for writing down the analytic form of $\mathcal{M}$ for a given process directly.

Let's work out explicitly how this arises in ϕ^4 theory.

We'll focus on the process

$$\phi(p_A) + \phi(p_B) \rightarrow \phi(p_1) + \phi(p_2)$$

The S-matrix we can write down directly from above!

$$\langle \vec{p}_1, \vec{p}_2 | T \left\{ \exp \left[-i \int d^4x \frac{\lambda}{4!} \phi_I^4(x) \right] \right\} | \vec{p}_A, \vec{p}_B \rangle$$

Expanding the exponential and looking at the first few terms, the leading term is just:

$$\langle \vec{p}_1, \vec{p}_2 | 1 | \vec{p}_A, \vec{p}_B \rangle = \sqrt{2E_1 2E_2 2E_A 2E_B} \\ \times \langle 0 | a_1 a_2 a_A^\dagger a_B^\dagger | 0 \rangle$$

$$= (2\pi)^3 2E_A \delta^3(\vec{p}_A - \vec{p}_1) \times (2\pi)^3 2E_B \delta^3(\vec{p}_B - \vec{p}_2) \\ + (1 \leftrightarrow 2)$$

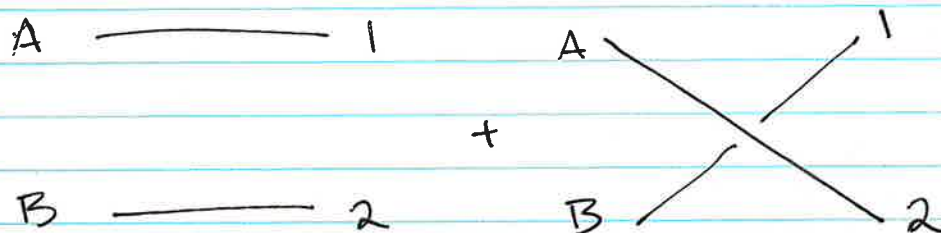
$$= \langle \vec{p}_1 | \vec{p}_A \rangle \langle \vec{p}_2 | \vec{p}_B \rangle + \langle \vec{p}_1 | \vec{p}_B \rangle \langle \vec{p}_2 | \vec{p}_A \rangle$$

Recall that we're using the Lorentz-invariant state normalization!

$$\langle \vec{p} | \vec{q} \rangle = (2\pi)^3 \cancel{8\pi} 2E_{\vec{p}} \delta^3(\vec{p} - \vec{q})$$

→ the δ -functions force the final state to be identical to the initial state.

This is clearly part of the "trivial" piece of the S-matrix. Diagrammatically, we can represent it as



Note that we have particles in the initial & final states (as opposed to the vacuum), which we denote w/ external lines.

This is in contrast to the Feynman diagrams for correlation functions, which ended on spacetime points, represented by dots.

Moving on to the next term, at $\mathcal{O}(\lambda)$ we have

$$\begin{aligned} \langle \vec{p}_1, \vec{p}_2 | T \left[-i \frac{\lambda}{4!} \int d^4x \phi^4(x) \right] | \vec{p}_A, \vec{p}_B \rangle &\rightarrow \text{(Wick's Theorem)} \\ &= \langle \vec{p}_1, \vec{p}_2 | \mathcal{N} \left[-i \frac{\lambda}{4!} \int d^4x \phi^4(x) + \text{all possible contractions} \right] | \vec{p}_A, \vec{p}_B \rangle \end{aligned}$$

At this stage, though — ~~not~~ we have uncontracted terms which do not vanish in general, since we no longer have vacuum states on the outside!

Annihilation operators a from $\phi(x)$ can annihilate an initial state particle, and similarly for final states.

For instance,

$$\begin{aligned}
 \phi^+(x) |\vec{p}\rangle &= \left[\int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_k}} a_{\vec{k}} e^{-ik \cdot x} \right] \left[\sqrt{2E_{\vec{p}}} a_{\vec{p}}^+ |0\rangle \right] \\
 &= \int \frac{d^3k}{(2\pi)^3} \sqrt{\frac{2E_{\vec{p}}}{2E_k}} e^{-ik \cdot x} \left(\underbrace{[a_{\vec{k}}, a_{\vec{p}}^+]}_{(2\pi)^3 \delta^3(\vec{k}-\vec{p})} + \cancel{a_{\vec{p}}^+ a_{\vec{k}}} \right) |0\rangle \\
 &= e^{-ip \cdot x} |0\rangle
 \end{aligned}$$

We can get lots of contributions from uncontracted ϕ operators this way. It's useful to extend our definition of a contraction for fields w/ initial/final states via:

$$\begin{aligned}
 \overline{\phi(x)} |\vec{p}\rangle &= e^{-ip \cdot x} |0\rangle \\
 \langle \vec{p} | \overline{\phi(x)} &= \langle 0 | e^{+ip \cdot x}
 \end{aligned}$$

Then we just need to evaluate all possible full contractions. Continuing w/ our $\mathcal{O}(x)$ term,

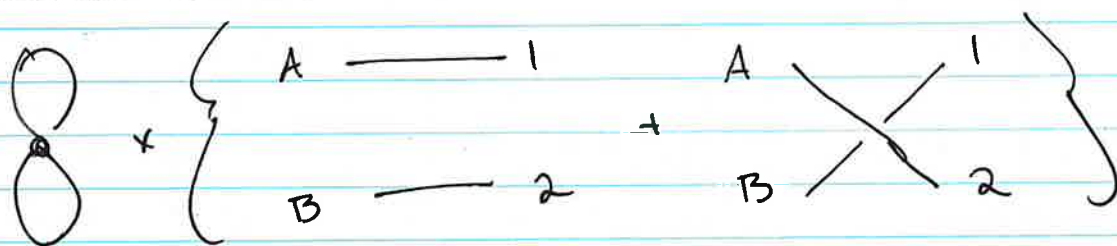
$$\langle \vec{p}_1 \vec{p}_2 | N \left[-i \frac{\lambda}{4!} \int d^4x \left(\underbrace{\phi_x^4}_{(1)} + 6 \underbrace{\overline{\phi_x} \phi_x \phi_x^2}_{(2)} + 3 \underbrace{\overline{\phi_x} \phi_x \overline{\phi_x} \phi_x}_{(3)} \right) \right] | \vec{p}_A \vec{p}_B \rangle$$

Now we can look at these terms in turn.

The final term is:

$$\begin{aligned}
 (3) &= 3 \left(\frac{-i\lambda}{4!} \right) \langle \vec{p}_1, \vec{p}_2 | \int d^4x \overline{\phi_x} \phi_x \overline{\phi_x} \phi_x | \vec{p}_A \vec{p}_B \rangle \\
 &= \frac{-i\lambda}{8} \left[\int d^4x D_F(x-x)^2 \right] \langle \vec{p}_1, \vec{p}_2 | \vec{p}_A \vec{p}_B \rangle
 \end{aligned}$$

→ This is just a vacuum bubble times the leading order piece from before:



This is another trivial contribution to the S-matrix, not of interest to us.

Now consider term (2):

$$\begin{aligned}
 (2) &= 6 \left(\frac{-i\lambda}{4!} \right) \langle \vec{p}_1, \vec{p}_2 | \int d^4x \overline{\phi_x} \phi_x N(\phi_x \phi_x) | \vec{p}_A \vec{p}_B \rangle \\
 &= 6 \left(\frac{-i\lambda}{4!} \right) \int d^4x D_F(x-x) \langle \vec{p}_1, \vec{p}_2 | (\phi_x^+ \phi_x^+ + 2\phi_x^- \phi_x^+ + \phi_x^- \phi_x^-) | \vec{p}_A \vec{p}_B \rangle \\
 &\quad \sim a a \quad \sim a^\dagger a \quad \sim a^\dagger a^\dagger
 \end{aligned}$$

After some algebra, we find only the $\phi_x^- \phi_x^+$ term gives a non-vanishing contribution.

We find:

$$(2) = G \cdot 2 \cdot \left(\frac{-i\lambda}{4!} \right) \sqrt{2E_1 2E_2 2E_A 2E_B} \int d^4x D_F(x-x) \\ \times \langle 0 | a_1 a_2 \left[\int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_k}} e^{ik \cdot x} a_k^\dagger \right] \left[\int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_q}} e^{-iq \cdot x} a_q \right] a_A^\dagger a_B^\dagger | 0 \rangle$$

This yields a ton of algebra to pick apart: $\langle 0 | a_1 a_2 a_k^\dagger a_q a_A^\dagger a_B^\dagger | 0 \rangle$

Introducing some shorthand:

$$\hat{\delta}_{ij} \equiv (2\pi)^3 \delta^3(\vec{p}_i - \vec{p}_j)$$

This yields:

$$\langle 0 | a_1 a_2 a_k^\dagger a_q a_A^\dagger a_B^\dagger | 0 \rangle \\ = \hat{\delta}_{k1} \hat{\delta}_{qA} \hat{\delta}_{B2} + \hat{\delta}_{k1} \hat{\delta}_{qB} \hat{\delta}_{A2} \\ + \hat{\delta}_{k2} \hat{\delta}_{qA} \hat{\delta}_{B1} + \hat{\delta}_{k2} \hat{\delta}_{qB} \hat{\delta}_{A1}$$

Now we can evaluate the integrals over d^3k, d^3q .
There are four terms. Focussing only on the first:

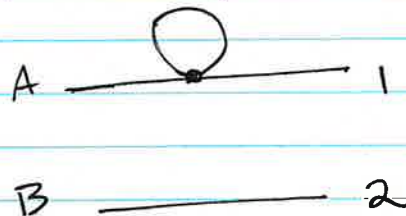
$$G \cdot 2 \cdot \left(\frac{-i\lambda}{4!} \right) \sqrt{2E_A 2E_B 2E_1 2E_2} \int d^4x D_F(0) \\ \times \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_k}} e^{ik \cdot x} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_q}} e^{-iq \cdot x} (2\pi)^3 \times \hat{\delta}_{k1} \hat{\delta}_{qA} \hat{\delta}_{B2}$$

$$= \frac{1}{2}(-i\lambda) \sqrt{\frac{2E_1 2E_2 2E_A 2E_B}{2E_1 2E_A}} (2\pi)^3 \delta^3(\vec{p}_B - \vec{p}_2) D_F(0)$$

$$\times \underbrace{\int d^4x e^{i(p_A - p_1) \cdot x}}_{= (2\pi)^4 \delta^4(p_A - p_1)}$$

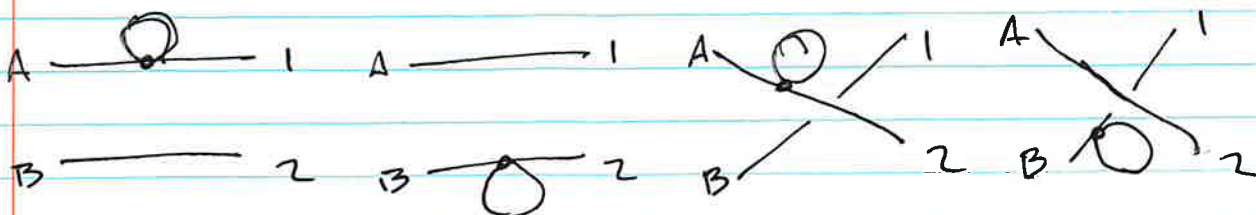
$$= \left\{ (2\pi)^3 \delta^3(\vec{p}_B - \vec{p}_2) \right\}_{(i)} \left\{ \frac{1}{2}(-i\lambda) D_F(0) (2\pi)^4 \delta^4(p_A - p_1) \right\}_{(ii)}$$

This is represented by the following diagram:



The first term (i) is just $\langle \vec{p}_2 | \vec{p}_B \rangle$. The second term has a loop from the contraction at the vertex, which gives $D_F(x-x) = D_F(0)$. It includes a factor of $-i\lambda$ from the vertex, and the $\int d^4x$ integral gave a momentum-conserv. δ -function.

The other 4 terms give us the full set:



We got this by using normal ordering and applying the commutation relations. We could also have written out the contractions:

$$\begin{aligned}
 (2) &= 6 \left(\frac{-i\lambda}{4!} \right) \langle \vec{p}_1, \vec{p}_2 | \int d^4x \overline{\phi}_x \phi_x N(\phi_x \phi_x) | \vec{p}_A \vec{p}_B \rangle \\
 &= 6 \left(\frac{-i\lambda}{4!} \right) \int d^4x D_F(0) \\
 &\quad \times 2 \times \left[\overbrace{\langle \vec{p}_1, \vec{p}_2 | \phi_x \phi_x | \vec{p}_A \vec{p}_B \rangle} + \overbrace{\langle \vec{p}_1, \vec{p}_2 | \phi_x \phi_x | \vec{p}_A \vec{p}_B \rangle} \right. \\
 &\quad \left. + \overbrace{\langle \vec{p}_1, \vec{p}_2 | \phi_x \phi_x | \vec{p}_A \vec{p}_B \rangle} + \overbrace{\langle \vec{p}_1, \vec{p}_2 | \phi_x \phi_x | \vec{p}_A \vec{p}_B \rangle} \right]
 \end{aligned}$$

where,

$$\begin{aligned}
 \overbrace{\langle \vec{p}_1, \vec{p}_2 | \phi_x \phi_x | \vec{p}_A \vec{p}_B \rangle} &= e^{-i\vec{p}_A \cdot x} e^{+i\vec{p}_1 \cdot x} \langle \vec{p}_2 | \vec{p}_B \rangle \\
 &= e^{-i(\vec{p}_A - \vec{p}_1) \cdot x} (2\pi)^3 \delta^3(\vec{p}_2 - \vec{p}_B)
 \end{aligned}$$

This gives the same result as before.

Once again, due to the momentum conservation δ -functions, these give (another) interesting contribution to the trivial part of the S-matrix.

Finally, we have the first term (1), at $\mathcal{O}(\lambda)$.

$$\begin{aligned}
 (1) &= \langle \vec{p}_1, \vec{p}_2 | N \left\{ -\frac{i\lambda}{4!} \int d^4x \phi_x \phi_x \phi_x \phi_x \right\} | \vec{p}_A, \vec{p}_B \rangle \\
 &= 4! \cdot \left(-\frac{i\lambda}{4!} \right) \int d^4x \langle \vec{p}_1, \vec{p}_2 | \overbrace{\phi_x \phi_x \phi_x \phi_x}^{} | \vec{p}_A, \vec{p}_B \rangle
 \end{aligned}$$

Here we've counted $4!$ equivalent contractions.

$$= (-i\lambda) \int d^4x e^{-i p_A \cdot x} e^{-i p_B \cdot x} e^{+i p_1 \cdot x} e^{+i p_2 \cdot x}$$

$$= (-i\lambda) (2\pi)^4 \delta^4(p_A + p_B - p_1 - p_2)$$

This is the interesting, non-trivial part of the S-matrix, corresponding to scattering. This S-matrix element is represented by the diagram:



Note that the matrix element has the general form anticipated above, $i\mathcal{M} (2\pi)^4 \delta^4(p_A + p_B - p_1 - p_2)$ w/

$$i\mathcal{M} = -i\lambda$$

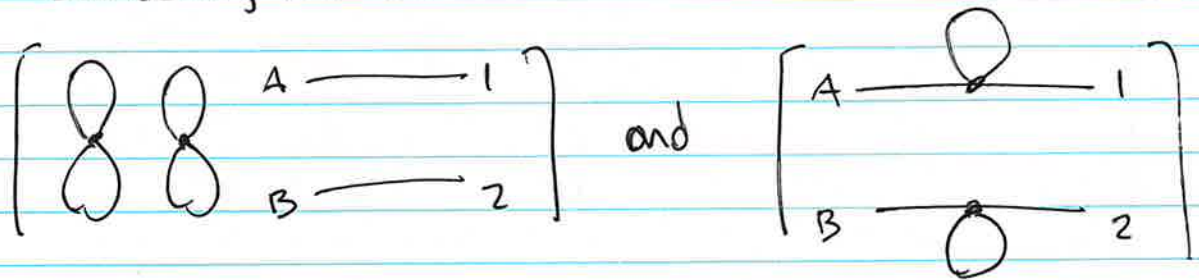
Looking back at our Feynman rule, we see $i\mathcal{M}$ is proportional to the rule for the vertex.

Let's press on to higher order corrections before we pick out the general rule. At $O(\lambda^2)$, the S-matrix element reads:

$$\begin{aligned} & \langle \vec{p}_1 \vec{p}_2 | T \left\{ \frac{1}{2!} \left[\int d^4x \left(\frac{-i\lambda}{4!} \right) \phi^4(x) \right] \left[\int d^4y \left(\frac{-i\lambda}{4!} \right) \phi^4(y) \right] \right\} | \vec{p}_A \vec{p}_B \rangle \\ &= \cancel{\langle \vec{p}_1 \vec{p}_2 |} \left(\frac{1}{2!} \right) \left(\frac{-i\lambda}{4!} \right)^2 \int d^4x d^4y \\ & \quad \langle \vec{p}_1 \vec{p}_2 | N \left(\phi_x^4 \phi_y^4 + \text{all possible contractions} \right) | \vec{p}_A \vec{p}_B \rangle \end{aligned}$$

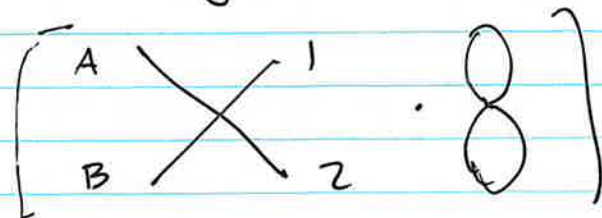
There are lots of possibilities now.

First, here are diagrams where the external lines are not connected, such as



These contribute to the trivial part of S, not the T-matrix.

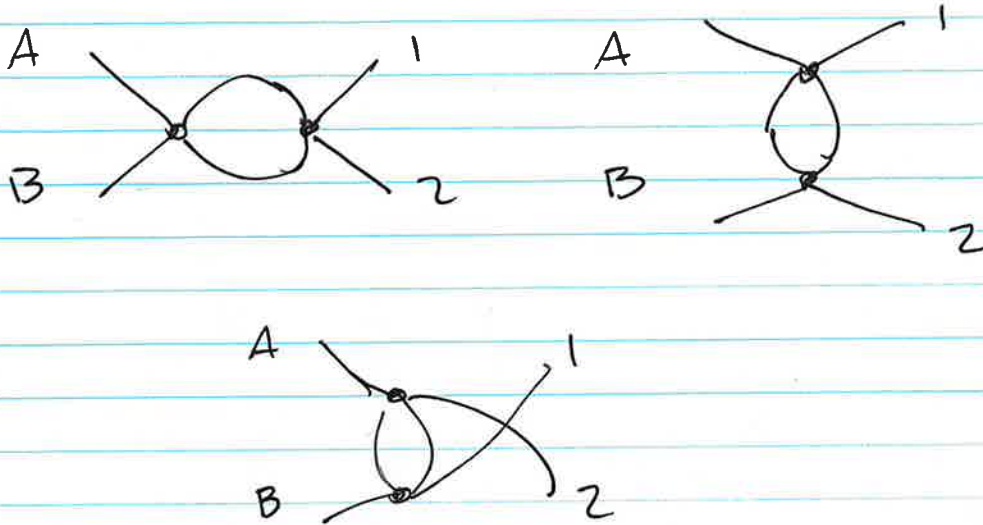
There are also diagrams of the form:



This gives a nontrivial contribution to the T-matrix, but also contains a disconnected vacuum bubble.

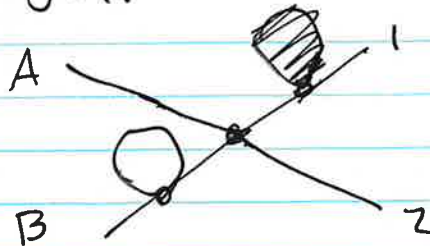
As w/ correlation functions, these vacuum bubbles exponentiate and give an overall phase factor, corresponding to a shift in the vacuum energy. Hence, we can ignore these diagrams, since they don't affect the scattering.

Next here are diagrams of the following kind:



These are genuine, $\mathcal{O}(\lambda^2)$ corrections to the T-matrix which must be accounted for in scattering processes.

Finally, here are amplitudes corresponding to diagrams of the following type:



Let's compute this! (We won't worry about the precise factor)

~~If corresponds~~ The corresponding amplitude is

$$\begin{aligned}
 & \sim (-i\lambda)^2 \int d^4x d^4y \langle \vec{p}_1, \vec{p}_2 | (\phi_x \phi_x \phi_x \phi_x) (\phi_y \phi_y \phi_y \phi_y) | \vec{p}_A, \vec{p}_B \rangle \\
 & = -\lambda^2 \int d^4x d^4y e^{-ip_A \cdot x} e^{-ip_B \cdot y} e^{ip_1 \cdot x} e^{ip_2 \cdot x} D_F(x-y) D_F(y-y) \\
 & = -\lambda^2 \int d^4x d^4y \int \frac{d^4k}{(2\pi)^4} \frac{i e^{-ik \cdot (x-y)}}{k^2 - m^2} \int \frac{d^4q}{(2\pi)^4} \frac{i}{q^2 - m^2} \\
 & \quad \times e^{-i(p_A - p_1 - p_2) \cdot x} e^{-ip_B \cdot y}
 \end{aligned}$$

$$= \lambda^2 \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2 - m^2} \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m^2} (2\pi)^3 \delta^4(p_A + k - p_1 - p_2) \times \delta^4(p_B - q)$$

Where we inserted the momentum rep. of the propagators.

Now, integrating over d^4k , we are forced to fix

$k = p_1 + p_2 - p_A = p_B$. But then, the denominator blows up:

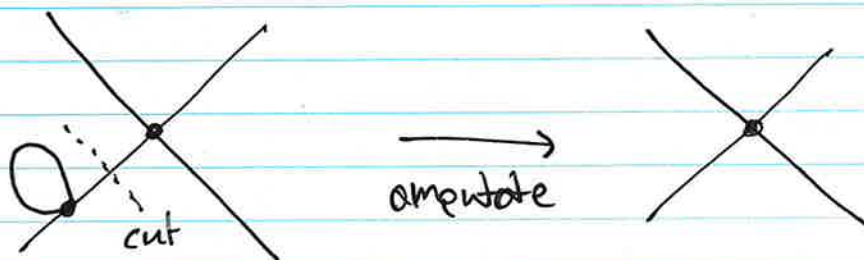
$$\frac{1}{k^2 - m^2} \Big|_{k=p_B} = \frac{1}{p_B^2 - m^2} \rightarrow \frac{1}{m^2 - m^2} \rightarrow \infty$$

These diagrams appear particularly problematic, but they actually have a clear, physical interpretation. Just as the vacuum bubbles represent evolution of the vacuum state from $|0\rangle$ to $|0\rangle$, diagrams w/ loops on external legs represent evolution of the single particle states $|p_0\rangle$ (free theory) to $|p\rangle$ (interacting theory)

Following this discussion, we're led to the notion of amputated diagrams:

Given a general diagram, we start at each external leg and find the last point at which the diagram can be cut by removing a single propagator, such that the rest of the diagram is separated from the leg.

E.g., from our last diagram!



With this, we can state the prescription for calculating scattering amplitudes — the non-trivial contributions to S-matrix elements:

$$i\mathcal{M} \cdot (2\pi)^4 \delta^4(p_A + p_B - \sum p_f)$$

$$= \left[\text{sum of all fully-connected, amputated Feynman diagrams, w/ } p_A, p_B \text{ incoming, } p_f \text{ outgoing} \right]$$

Fully connected diagrams are those w/ no vacuum bubbles, and all external legs connected to each other.

The Feynman Rules for scattering amplitudes in ϕ^4 theory in position space are:

1. For each propagator,

$$\begin{array}{c} \bullet \\ x \end{array} \text{---} \begin{array}{c} \bullet \\ y \end{array} = D_F(x-y)$$

2. For each vertex,

$$\begin{array}{c} \diagup \\ \bullet \\ \diagdown \end{array} = (-i\lambda) \int d^4x$$

3. For each external line

$$\begin{array}{c} \text{the rest of the} \\ \text{diagram} \end{array} \rightarrow \begin{array}{c} \diagup \\ \bullet \\ \diagdown \end{array} \begin{array}{c} \leftarrow \\ p \end{array} = e^{-ip \cdot x}$$

4. Divide by the symmetry factor.



In practical calculations, it's much more common to use the momentum-space Feynman rules, to obtain the momentum space representation of scattering amplitudes.

Here, after inserting the FT of all propagators, we will always be left with a single δ -function of momenta, ~~where~~ which enforces total four-momentum conservation.

Then, we can compute $i\mathcal{M}$ directly:

$$i\mathcal{M} = \left(\text{sum of all fully connected, amputated diagrams} \right)$$

w/ the momentum space Feynman rules:

1. For each propagator,

$$\overrightarrow{p} = \frac{i}{p^2 - m^2 + i\epsilon}$$

2. For each vertex,

$$\begin{array}{c} \swarrow \quad \nearrow \\ \bullet \\ \nwarrow \quad \searrow \end{array} = -i\lambda$$

3. For each external line,

$$\begin{array}{c} \rightarrow \quad \leftarrow \end{array} = 1$$

4. Impose momentum conservation at each vertex

5. Integrate over unconstrained loop momenta: $\int \frac{d^4 k}{(2\pi)^4}$

6. Divide by the symmetry factor.

This gives us our final prescription for computing scattering amplitudes in an interacting theory with Feynman diagrams!

We don't have time to delve into it here, but a more rigorous justification for the prescription that relates scattering amplitudes to Feynman diagrams is given by the Lehmann, Symanzik, Zimmermann (LSZ) reduction formula.

The statement relates correlation functions to S-matrix elements:

$$\prod_{i=1}^n \int d^4 x_i e^{i p_i \cdot x_i} \prod_{j=1}^m \int d^4 y_j e^{i q_j \cdot y_j}$$

$$\times \langle 0 | T \{ \phi(x_1) \dots \phi(x_n) \phi(y_1) \dots \phi(y_m) \} | 0 \rangle$$

$$\approx \left(\prod_n \int d^4 x_i e^{i p_i \cdot x_i} \right) \left(\prod_m \int d^4 y_j e^{i q_j \cdot y_j} \right) \langle 0 | T \{ \phi(x_1) \dots \phi(x_n) \phi(y_1) \dots \phi(y_m) \} | 0 \rangle$$

LSZ identifies the S-matrix element as the ~~multi-particle~~ residue of the multi-particle pole of the Fourier transform of the correlation function. The symbol $\sim$ is used to mean that the poles on each side are the same, though the RHS has additional finite terms.

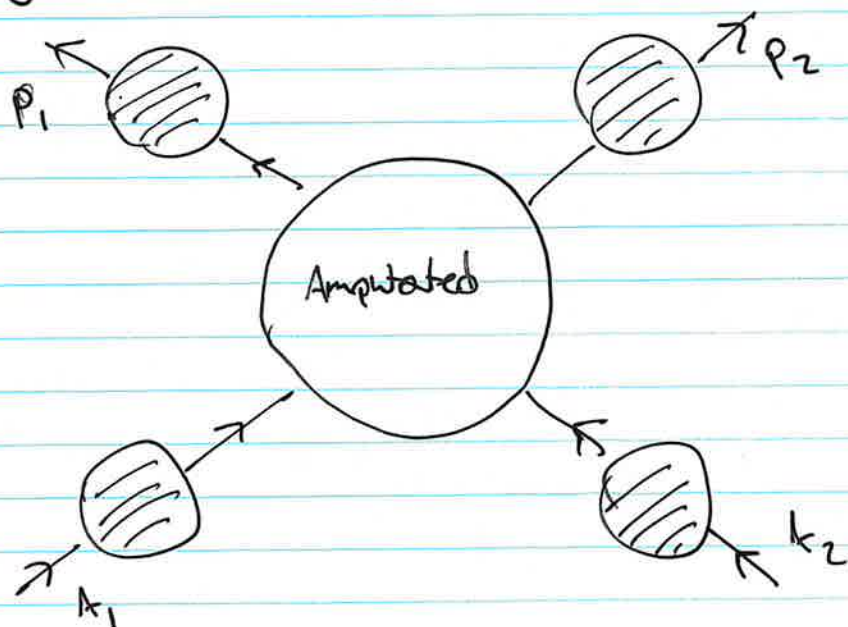
The new quantity Z is known as the field strength renormalization. It is the residue of the ~~pole~~ single-particle pole in the two-point correlation function.

At leading order, $Z=1$, but it receives corrections of $O(\hbar)$ once loop diagrams are accounted for.

The LSZ reduction formula can be represented in terms of Feynman diagrams.

The point is that the connected but not fully connected diagrams on the LHS do not have the correct singularity structure w/ a product of n -poles to contribute on the RHS.

E.g., for the 4-point correlator, the fully connected diagrams have the general form:



The corrections to each external leg can be summed as follows. Denote by $-iM^2(p^2)$ the sum of all "one-particle irreducible" or "1PI" diagrams entering the scalar propagator:

$$-iM^2(p^2) = \text{diagram with a tadpole} + \text{diagram with a bubble}$$

$$\text{diagram with a tadpole} + \dots = \text{diagram with a bubble}$$

A diagram is 1PI if it cannot be split into two diagrams by removing a single line.

(e.g., $\text{diagram with two bubbles}$ is not 1PI)

The exact propagator can then be written as a geometric series and summed:

$$\text{diagram with a bubble} = \text{diagram with a tadpole} + \text{diagram with a bubble} + \dots$$

$$= \frac{p^2 - m_0^2}{i} + \left(\frac{p^2 - m_0^2}{i} \right) (-iM^2(p^2)) \left(\frac{p^2 - m_0^2}{i} \right) + \dots$$

$$= \frac{p^2 - m_0^2 - M^2(p^2)}{i} \quad (\text{w/ } m_0 \text{ the "bare mass"})$$

A positive contribution to $M^2(p^2)$ corresponds to a positive shift to the squared scalar particle mass.

Expanding the resummed propagator about the physical pole, we see that each external leg of the four-point amplitude contributes:

$$\frac{p^2 - m_0^2 - M^2(p^2)}{z} \sim \frac{p^2 - m^2}{z} + (\text{regular})$$

Thus, the sum of all diagrams contains a product of four poles:

$$\frac{p_1^2 - m^2}{z} \cdot \frac{p_2^2 - m^2}{z} \cdot \frac{k_1^2 - m^2}{z} \cdot \frac{k_2^2 - m^2}{z}$$

Comparing the coefficients of this product of poles:

$$\langle p_1 p_2 | S | k_1 k_2 \rangle = (\sqrt{z})^4$$


This generalizes to an arbitrary $2 \rightarrow n$ process, w/ a factor of $(\sqrt{z})^{n+2}$ factor out front.

We'll defer more discussion of these issues (including proper renormalization theory) to a QFT course.